LARGE DISPLACEMENT ANALYSIS OF ELASTOPLASTIC BENDING OF NON-UNIFORM BEAM

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ABSTRACT

The aim of this work is to simulate the elastoplastic behavior in large displacements of a beam of non-uniform section subjected to flexion. First, a mathematical formulation of the elastoplastic behavior based on the moment-curvature relation in the framework of the classical beam theory is developed where the non-uniformity of the section is taken into consideration. The curvature is then evaluated numerically by means of the Newton-Raphson method. Then a formulation in large displacements is put in place. The geometric nonlinearity governed by differential equations is analyzed. The integration method of Euler is favored to solve the problem and thus determine the deflection of the beam. The results of the elasticity simulation are compared with other results of the literature and a good coherence was found in the light of which the approach was applied in elastoplasticity.

KEYWORDS: *Bending; curvature; elastoplastic ; large displacements*

1.0   INTRODUCTION

The elastoplastic analysis of beams in large displacements has been the subject of intense research in recent years because of its great importance in different engineering applications (Hong, 2000; Pandit & Srinivasa, 2016) including civil engineering and particularly in the estimation of performances structures in the context of seismic risk (He & Zhong, 2012), mechanical engineering, biomechanics and aeronautics (Al-Sadder & Al-Rawi, 2006). It is therefore justified to take an interest in the problem with the aim of proposing approaches able to predict the behavior of these beams in the face of mechanical stresses.

In elasticity, the behavior is often translated by a linearity relation between the stresses and the strains. However, plastic behavior is generally considered

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non-linear: it is a so-called material non-linearity. In bending, the curvature moment relation is well placed to represent the mechanical behavior (Fard, Chattopadhyay & Liu, 2012; Kwak & Kim, 2002; Royer-Carfagni, 2001). In turn, this relationship is linear in elasticity but nonlinear in plasticity. Moreover, if we consider that the displacements and the rotations are large, another geometric non-linearity intervenes. Thus, the problem becomes complicated and often requires the use of numerical methods for its resolution. Reference is made to the reference (Dado & Al-Sadder, 2005) for a brief history of the methods used to solve the problem of the bending of cantilever in large displacements. There are four main approaches, the first of which is based on elliptic integration, the second uses numerical integration, the third uses the finite element method in coupling with the Newton-Raphson method, and the fourth uses the method of the finite differences in coupling with the Newton-Raphson method.

Euler-Bernoulli's theory of flexion, which applies not only to model elastic behavior but also to describe the inelastic behavior of beams (Bazant & Cedolin, 2003), makes it possible to establish a relation between the flexural curvature and the transverse displacement of the beam. The curvature is in turn connected with the bending moment by a function which may be linear or nonlinear. In linear elasticity, different methods have been applied to evaluate the deflection in large displacements. Reference is made to the references (Ang, Wang & Low 1993; Beléndez, Neipp & Beléndez 2002; Dado & Al-Sadder, 2005; Al-Sadder & Al-Rawi, 2006; Wang, Chen & Liao, 2008; Chen, 2010; Mutyalarao, Bharathi & Rao, 2010; Shvartsman, 2013) which have taken into account various aspects including load types and very varied geometries. In the past literature, (Lewis & Monasa, 1981; Lewis & Monasa, 1982; Lee, 2002), the focus was on solving the problem of bending in large displacements of nonlinear elastic beams subjected to a concentrated load. In another plane, the article by Baykara, Guven and Bayer (2005) focused on the flexion in large displacements of bimodular nonlinear elastic materials. The tension behavior in this case is different from that in compression, while an analytical solution to the problem was successfully established. In the context of plastic behavior, the work by Fried1 (1985), He and Zhong (2012), Xi, Liu and Li (2012) as well as Pandit and Srinivasan (2016) have proposed approaches in large displacements concerning the elastic perfectly plastic and elastoplastic behavior with hardening.

Hence, based on the work reported in the literature, in this study, the focus is in the modeling and simulation in large displacements of a beam of elastoplastic behavior with linear hardening. The mechanical behavior is therefore assumed to be bilinear. Material and geometric nonlinearities are then taken into account. In addition, beams with non-uniform cross-sections are considered.

2.0 MATHEMATICAL FORMULATION

2.1 Constitutive law

The beam in question is considered to have elastoplastic behavior with linear hardening. Below a stress threshold (elastic limit), the response of the material is of linear elastic nature. Once this threshold is exceeded, the response is elastoplastic with linear hardening as illustrated in Figure 1. The constitutive law is given by an expression in Equation (1):

$$\begin{cases} \sigma = E\varepsilon & \varepsilon = \varepsilon_e & \sigma \leq \sigma_e \\ \sigma = \sigma_e + H\left(\varepsilon - \dfrac{\sigma_e}{E}\right) & \varepsilon = \left(\varepsilon_e + \varepsilon_p\right) & \sigma > \sigma_e \end{cases} \tag{1}$$

Where σ and σ_e denote respectively the stress and the elastic limit. ε_e, $\varepsilon_{\_p}$ and ε are respectively the elastic, plastic and total strains. E and H represent respectively the Young's modulus and tangent modulus.
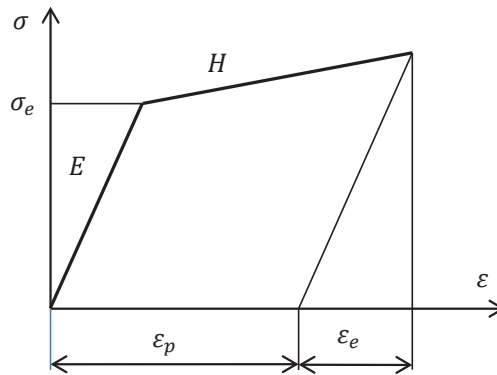

Figure 1 Elastoplastic behavior

2.2 Analysis of elastoplastic bending

Considering a straight cantilever beam of length L and rectangular section ($b \times 2h$) fixed at one end and subjected to a concentrated load P of the other free end as shown in Figure. 2. The applied force generates a zero bending

moment at the free end and increases as it moves towards the recess where it takes its maximum value. Since the stress produced is non-uniform in the direction of the length of the beam, two behavioral domains are distinguished: an elastic domain concerning the sections of the beam where the stress is below the elastic limit; and an elastoplastic domain in the sections where this limit is exceeded. The elastic boundary coincides with the coordinate corresponding to the plastification of the first fiber of the beam. It is therefore time to define a critical moment value M_e capable of generating an elastoplastic deformation. By designating I the quadratic moment of inertia of the section, the threshold moment is given by an expression in Equation (2) as follow:
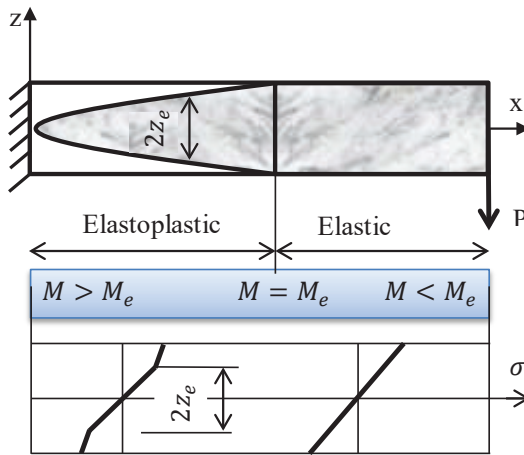
(2)

Figure 2 Boundary elastic / elastoplastic

Figure 2 shows that the sections of the beam in the elastic domain are of perfectly elastic behavior while those of the elastoplastic domain are partially plasticized. Indeed, due to the non-uniformity of the stresses in the direction of the height of the beam, two zones of different behaviors are observed within the same section in the elastoplastic domain. The elastic boundary z_e according to the height is obtained from the formula given in Equation (3) as follows:

$$z_e = \frac{h\chi_e}{\chi} \tag{3}$$

Where χ_e represents the curvature generated by the threshold moment M_e.

2.2.1 *Elastic field*

Elastic analysis concerns sections where the bending moment is below the plasticity threshold. The laws of linear elasticity in flexion are then applied. The bending curvature is easily determined from the equilibrium equation given in Equation (4) as follows:-

$$\chi = \frac{M}{EI} \tag{4}$$

2.2.1 *Elastoplastic field*

The elastoplastic analysis concerns all the sections of the elastoplastic domain. Two zones of different behaviors coexist within the same section. Thus, two different formulas of constraint govern these sections are as given in Equations (5) and (6) as follow:

$$\begin{cases} \sigma = E\varepsilon & z \leq z_e \\ \sigma = \sigma_e + H\left(\varepsilon - \frac{\sigma_e}{E}\right) & z > z_e \end{cases} \tag{5}$$

The equilibrium of the sections is governed by the equation:

$$M = 2b\left(\int_0^{z_e} E\varepsilon z \, dz + \int_{z_e}^h \left(\sigma_e + H\left(\varepsilon - \frac{\sigma_e}{E}\right)\right) z \, dz\right) \tag{6}$$

By integrating and replacing z_e by its formula, the following expression in Equation (7) is attained:

$$-\left(\frac{1}{2} + \frac{H}{E}\right)\left(\frac{\chi_e}{\chi}\right)^2 + \left(\frac{H}{E}\right)\frac{\chi}{\chi_e} + \frac{3}{2}\left(1 - \frac{H}{E}\right) - \frac{M}{M_e} = 0 \tag{7}$$

Since the equation is nonlinear, a numerical scheme is needed to determine the bending curvature in the elastoplastic domain. It is chosen the iterative method of Newton-Raphson.

It should be mentioned that in the case of elastic perfectly plastic behavior, Equation (7) reduces to a simpler expression as given in Equation (8) and does not require any numerical analysis for its resolution.

$$-\left(\frac{1}{2}\right)\left(\frac{\chi_e}{\chi}\right)^2 + \frac{3}{2} - \frac{M}{M_e} = 0 \tag{8}$$

Figure 3 shows the moment curvature relationship in the case of elastic, elastic perfectly plastic and bilinear elastoplastic behaviors. It shows a

linear configuration corresponding to the elastic behavior while a nonlinear appearance reigns over the elastoplastic behavior, whether it is perfect or hardening.
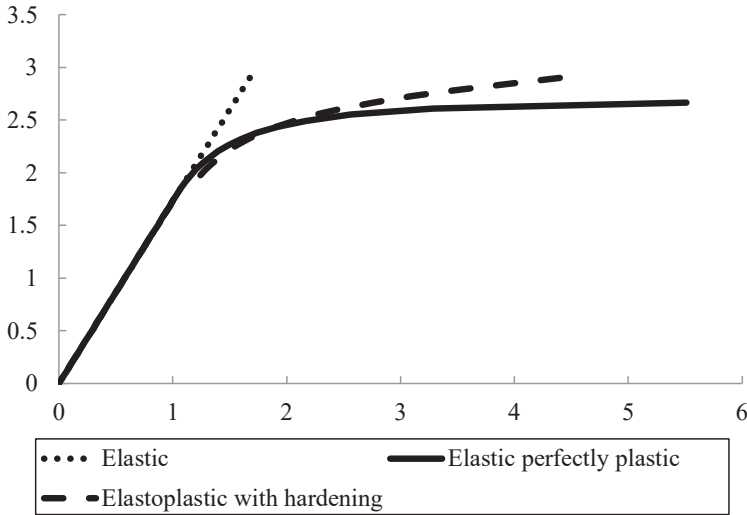

Figure 3 Moment curvature relationship

2.3   Elastoplastic analysis of beams with non-uniform sections

In the case of beams with non-uniform sections, the quadratic moment of the section being variable along the beam, the threshold moment also varies from one section to the other. In other words, the appearance of the plastic zones can take place at any section and not necessarily at the sections where the moment of flexion is greater. Thus, in order that the response of the beam is purely elastic in all the sections of the beam, the following condition must be satisfied, as given in Equation (9):

$$M(x) \leq \frac{I(x)\sigma_e}{h} \tag{9}$$

Considering a beam of circular cross-section whose diameter is controlled by a function $D(x)$, the function of the diameters necessary so that there is no appearance of plastic zones along the beam is expressed by Equation (10) as follow:

$$D(x)^3 \geq \frac{32}{\pi\sigma_e} M(x) \tag{10}$$

Equation (10) could serve as a criterion for the resistance of the beam. It can be seen that the necessary resistive diameters are functions of the loading and of the plasticity threshold stress. The analytical determination of these diameters is often a complicated task. The complexity is relative to the load and the function itself. However, numerical analyzes for its determination are always available.

In the case of a cantilever beam of linearly variable cross-section in the longitudinal direction, with initial and final diameters designated respectively by D_0 and D_1, the function of the diameters is expressed by Equation (11) as follow:

$$ \hspace{8cm} (11)$$

As illustrated in Figure 4, and differently to beams having uniform cross-sections, the appearance of the plastic zones can take place first either at the embedding or elsewhere. It depends on the intensity of the linearity of the function of the diameters. Moreover, its evolution is relative to the intensity of the loading.
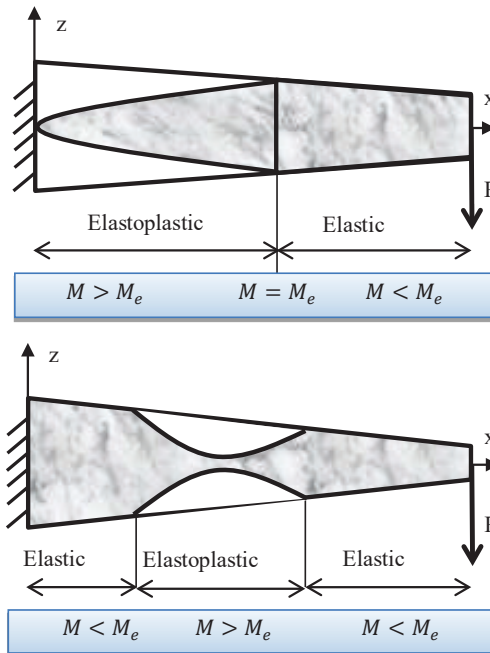

Figure 4 Appearance of plastic areas in a non-uniform section beam

It should be mentioned that it is possible to proceed in the same way for problems with different geometries and / or loadings. However, the complexity of the problem also depends on the complexity of the geometries and the loading.

2.3 Large deflection formulation

The problem of bending of beam with large deflection is shown in Figure 5. According to the classical beam theory based on Euler-Bernoulli's kinematic hypotheses, the straight sections of a beam remain straight and perpendicular to the average fiber. Thus, the transverse shear effect is assumed to be negligible. The bending curvature, as given in Equation (12) is assimilated to the derivative of the angle of rotation θ.

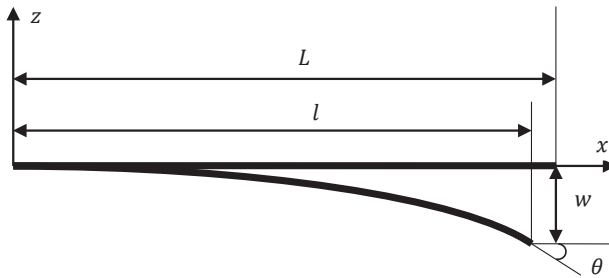$$\chi = \frac{\partial \theta}{\partial x} \tag{12}$$

Figure 5 Large deflection cantilever beam

In large displacements and large rotations, the second derivative of the curvature is no longer assimilated to the deflection w. However, it is connected with it Equation (13) as follow:

$$\chi = \frac{\frac{\partial^2 w}{\partial x^2}}{\left(1 + \left(\frac{\partial w}{\partial x}\right)^2\right)^{3/2}} \tag{13}$$

Equations (13) have been the subject of numerous studies, the majority of which assumes that the displacements are so small that the magnitude $(\partial w/\partial x)^2$ 2 representing the square of the arrow is negligible (Timoshenko, 1968; Da Silva , 2006) . This simplifying hypothesis can be expressed as in Equation (14):

$$\chi = \frac{\partial^2 w}{\partial x^2} \tag{14}$$

Equation (14) can be solved analytically by the direct integration of the curvature under the. However, equation (13) often requires numerical methods. In the literature (Chen, 2010; Ang et al., 1993) there are many methods of resolutions. Assuming that the neutral axis does not undergo deformation, the length of the curve of the beam $s(l)$ is calculated Equation (15) as follow:

$$s(l) = \int_0^l \sqrt{1 + (dw/dx)^2} dx \tag{15}$$

Ang et al (Ang et al., 1993) proposes a numerical approach of resolution by setting $y = (\partial w/\partial x)$. Thus, the equations which govern the bending of the cantilever beam in large displacements are summarized in Equation (16) as follow :

$$\begin{cases} \frac{\partial y}{\partial x} = \chi(1 + y^2)^{\frac{3}{2}} \\ \frac{\partial w}{\partial x} = y \\ \frac{ds}{dx} = \sqrt{1 + y^2} \end{cases} \tag{16}$$

Since the bending curvature is evaluated both in the elastic and in the elastoplastic domains, appropriate numerical schemes are needed to evaluate the deflection. The equations above correspond to differential equations. There are different methods to solve them. Euler's method of explicit integration is favored in this study. This choice is justified by the simplicity of implementation and the reduced computing time without the results being divergent. To do this, the length of the beam is divided into $(N\text{-}1)$ elements defined by Δx with N nodes. The coordinates of the nodes are designated by $(x_1, x_2, x_3, \ldots, x_N)$. The discretization of equations (16) is given by an expression in Equation (17) as follow :

$$\begin{cases} i = 1,2,3, \ldots, N \\ x_1 = y_1 = w_1 = s_1 = 0 \\ x_{i+1} = x_i + \Delta x \\ y_{i+1} = y_i + \Delta x \left(\chi_i (1 + y_i^2)^{\frac{3}{2}} \right) \\ w_{i+1} = w_i + \Delta x(y_i) \\ s_{i+1} = s_i + \Delta x(\sqrt{1 + y_i^2}) \end{cases} \tag{17}$$

3.0   RESULTS AND DISCUSSION

The results of the simulation focus on the validation of the method adopted in elasticity. The method is subsequently applied in elastoplasticity.

3.1   Elastic validation

We take up the example applied in the work of Chen (2010). A 0.2 *m* long cantilever beam has a non-uniform circular section with a diameter of 0.002 *m* at the fixed end and a diameter of 0.0002 *m* at its free end. The beam is considered to have a linear elastic behavior with a Young's modulus $E=1.2\times 10^{11}$ *Pa* Pa. It is loaded vertically at its free end by a concentrated force with 4 loading levels $P = (0.1.0.2.0.5,1)N$. The deflection of the beam is illustrated in Figure. 6. The comparison of the results of the elastic deflections is presented in Table 1. It is noted that the results are so close that the present approach is validated.

Table 1 Correlation of elastic deflections of the present work with those of Chen (2010)

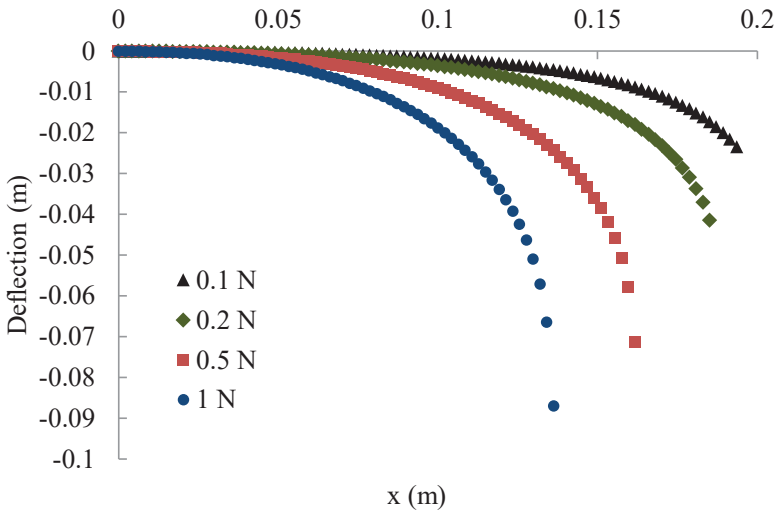| Force (N)		0.1	0.2	0.5	1
w/L	Chen	0.124	0.203	0.328	0.430
	Present work	0.117	0.207	0.356	0.434
l/L	Chen	0.977	0.941	0.860	0.779
	Present work	0.98	0.94	0.84	0.72

Figure 6 Elastic deflection

3.2   Application in elastoplasticity

Considering a bilinear elastoplastic material, the moment curvature relation is nonlinear. Consider a beam of length *L* with a non-uniform circular cross-section with the largest diameter D_0 at the fixed end while the smallest D_1 at its free end. The parameters of the material of which it is composed (Pandit & Srinivasa, 2016) are represented by the Young modulus *E*=2.1 ×10^{11} *Pa* and the tangent modulus *H*=0.25*E*.

3.2.1   *Appearance and evolution of the plastic zone*

To demonstrate the influence of the non-uniformity of the section on the appearance of the plastic zone, two geometric configurations have been proposed as shown in Figure 7. It can be seen that in the configuration, the plastic area did not appear at either end of the beam. Indeed, the diameters at the ends are so large that the beam resists there without this being the case throughout the beam. However, in the configuration *b*, where the diameter of the embedded end of the beam is so small that the plasticity appears therein without it being at the other free end.

The usefulness of the non-uniformity of the section makes it possible to save material by recommending the optimal geometry. Thus, it is possible to provide a beam of the same volume but with a different geometry than that of the configuration a (D_0 = 0.0055, D_1 = 0.0025) without there being any plastic zone.
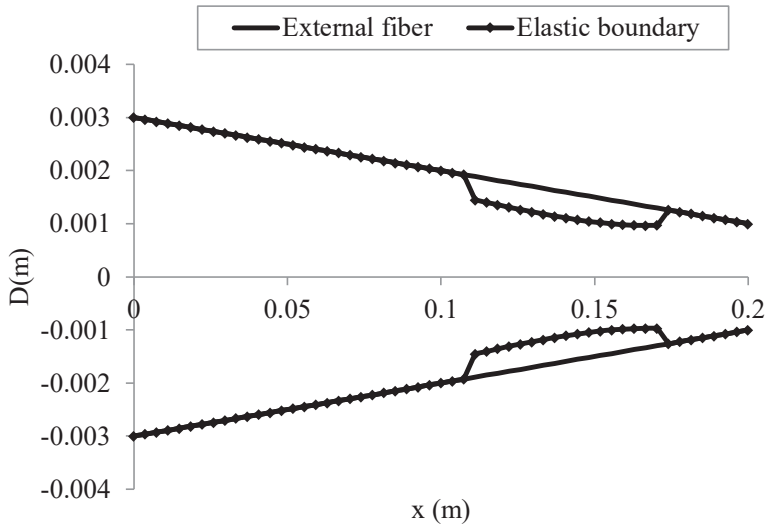
(a) $(D_0=0.006, D_1=0.002)$m
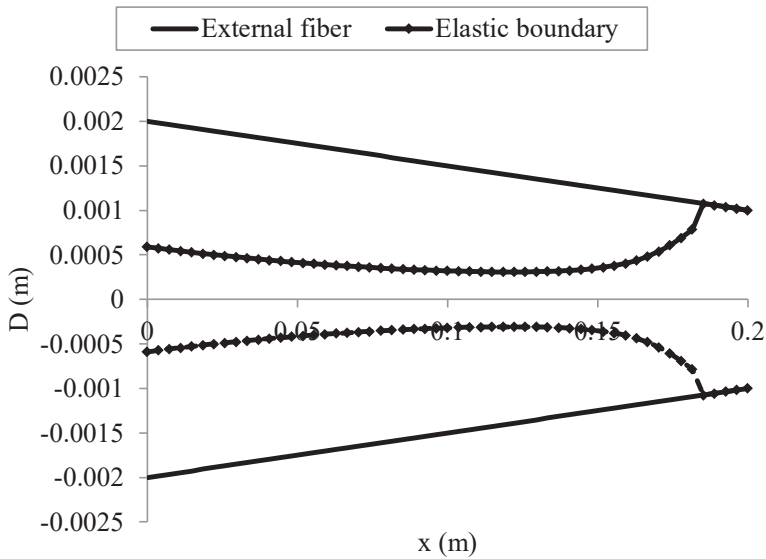
(b) $(D_0=0.004, D_1=0.002)$m

Figure 7 Appearance of the plastic zone in the beam

3.2.2 *Influence of loading*

The beam is subjected to three load levels of 10, 15 and 20 *N*, respectively. The geometry of the beam is maintained in the configuration ($D_0 = 0.004$, $D_1 = 0.002$) *m*. Figure 8 shows the curvature generated in the beam by the three loading

levels. Since it is zero at the free end, it evolves nonlinearly as a function of the bending moment and the geometry of the beam as it moves towards the fixed end. Although the bending moment is maximal at the level of the fixed end, the curvature is not there. This is explained by the increase in the diameter allowing it to be resistant. The moment curvature relationship in the beam is translated by the curves in Figure 9. The non-uniformity of the section influences considerably on the shape of the curves. Thus, different curves can be provided by varying the cross-section along the beam. The evolution of the plastic zone under the three loads is illustrated in Figure 10. It can be seen that the loading influences significantly on the plastic zone, both according to the length and the height of the beam.

Figure 11 illustrates a confrontation of the elastic deflections with the elastoplastic deformations of the beam under the three loads. The elastoplastic déflections are larger than the elastic deflections under the highest loadings. This explains why plasticity is very sensitive to loading.
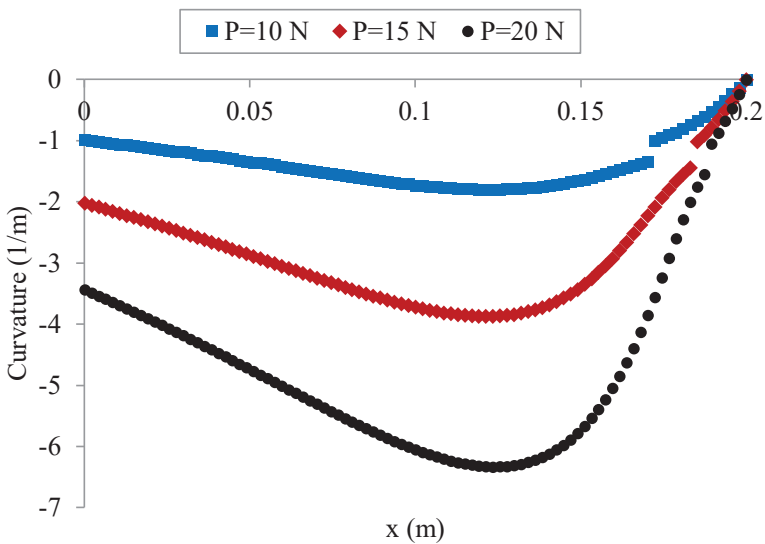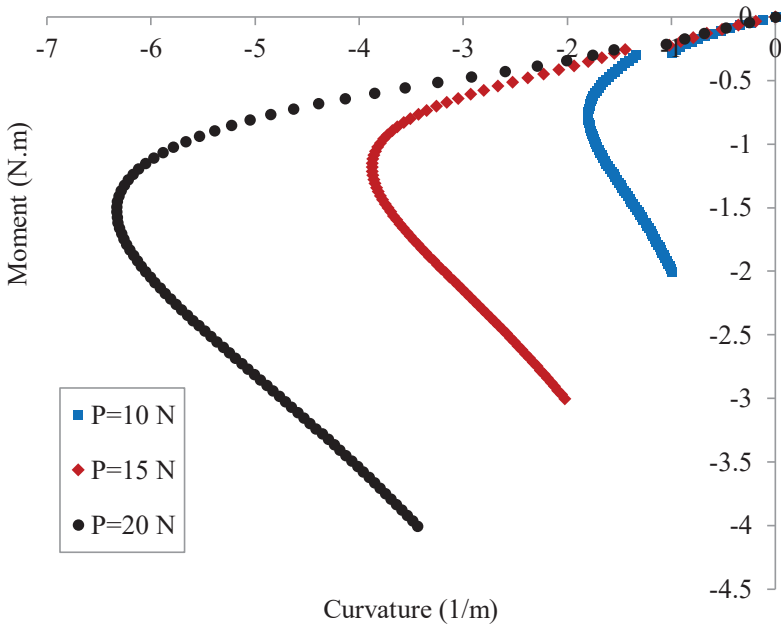

Figure 8 Curvature graph in the beam

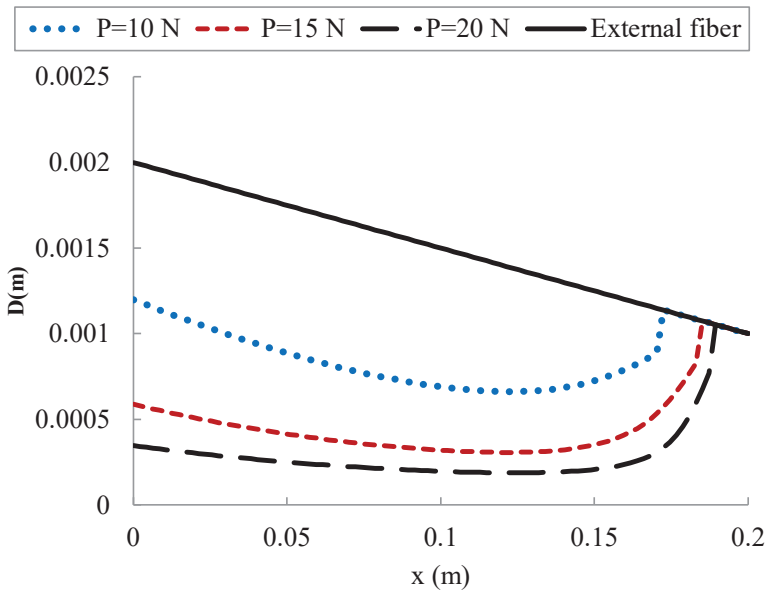**Figure 9** Moment curvature relationship in the beam

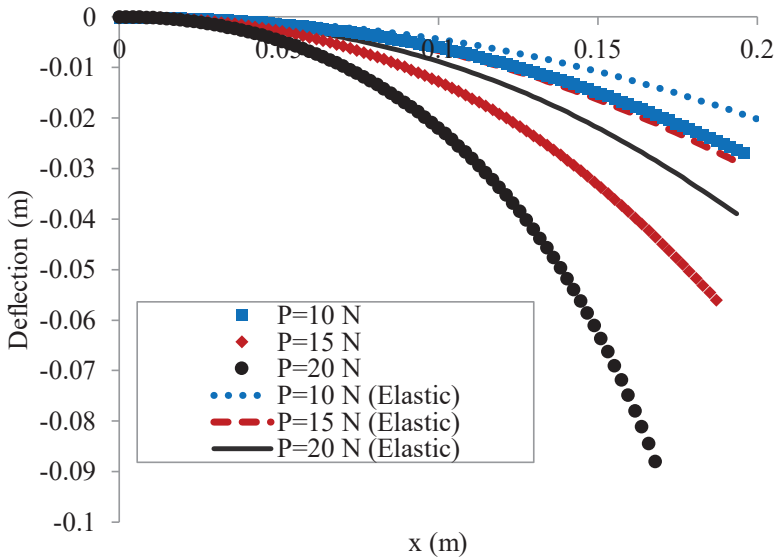**Figure 10** Evolution of the plastic area

Figure 11 Evolution of the deflection of the beam

4.0   CONCLUSION

The present work analyzes the elastoplastic behavior of a beam of non-uniform cross-section in large displacements. The mathematical equations controlling the behavior and the geometry of the beam were formulated. From a material point of view, since the moment curvature relation is nonlinear, Newton-Raphson's method has been favored for its resolution. On the geometrical plane, the formulation in large displacements leads to ordinary differential equations solved with the integration method of Euler. This analysis has demonstrated the effect of the non-uniformity of the section of the beam on the appearance of plastic zones where a resistance criterion has been established.

The results of the elastic analysis of beams with non-uniform sections in large displacements were compared with others in the literature. The good coherence observed made it possible to validate the approach. In elastoplasticity, only a presentation and interpretation of the results was discussed. Validation by experimental tests remains in perspective.SI units must be used throughout the manuscript.

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