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THE DERIVATION OF A DESIGN MATRIX FOR A FOOT-MOUNTED INERTIAL PEDESTRIAN NAVIGATION USING INVARIANT OBSERVER APPROACH

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Abstract— This paper describes a derivation of a design matrix for a foot-mounted inertial pedestrian navigation. The design matrix sometimes known as state space propagation matrix, or transition matrix, that propagate the modelled states over time. An inertial sensor is assumed to be strapped tightly on the foot of a pedestrian, and therefore the measurements obtained are assumed to be highly correlated with the movement of a foot. This permit the use of velocity update whenever the foot is on the ground. The design matrix is then derived using the Invariant Extended Kalman Filter (IEKF) framework. The navigation state is represented as an element of the matrix Lie group of double direct isometries, which is a mathematical description of the space in which the pedestrian moves, including position, velocity, and attitude. The model also incorporates accelerometers and rate-gyros biases, which are common in inertial sensors. A comparison with the design matrix derived from the standard Extended Kalman Filter (EKF) are made, and will be shown unvarying with attitude estimates, which is an improvement over the standard EKF.

I. Introduction

The Extended Kalman Filter (EKF) is a widely used mathematical tool for estimating key variables and correcting errors [1]. For an inertial pedestrian navigation, the EKF typically involves a 15-state model, organized in a 3-by-3 matrix, encompassing position, velocity, orientation, accelerometer bias, and gyroscope bias. This model is used to predict how these states change over time.

When a new measurement is received from an external sensor, it updates the estimated states. The error-based EKF approach compares the predicted states with the latest sensor measurement to calculate the error. By giving weight to this error, it's used to adjust the predicted states, thereby refining or updating them. This loop of prediction and correction based on new measurements makes the EKF a powerful estimation tool.

The design matrix is a mathematical framework that outlines how states change over time within the context of state estimation using the Extended Kalman Filter (EKF), also known as the Multiplicative Extended Kalman Filter (MEKF). In the EKF, the state vector is continuously predicted and updated based on data from external sensors. The design matrix plays a critical role in modeling the relationship between the state vector and the sensor measurements, allowing the state vector to be projected forward in time.

Typically, the design matrix is derived from the underlying physical model of the system being estimated. For example, in inertial navigation, the design matrix establishes the relationship between acceleration, velocity, and position states, illustrating how they evolve over time. It also incorporates terms to represent sensor biases and other sources of error.

Recent studies in [2], [3], and [4] introduced a new method for state estimation on Lie groups using the Invariant Extended Kalman Filter (IEKF). The Lie group framework allows modeling a system's state that evolves on a nonlinear manifold, offering a broader scope than the traditional Kalman filtering approach, which typically uses Euclidean space. Lie groups [10] are mathematical constructs that capture a system's symmetry properties. Essentially, a Lie group is a collection of transformations that can be smoothly parameterized and that also form a differentiable manifold. In other words, it's a group with a smooth structure. When a system is invariant under a Lie group's transformations, it means the system's characteristics remain consistent when these transformations are applied. The IEKF is a state estimation algorithm that operates on Lie groups. It is a nonlinear filter that can handle a wide range of nonlinear and non-Gaussian systems, making it well-suited for many real-world applications, including inertial pedestrian navigation [5].

In this article, the Invariant Extended Kalman Filter (IEKF) algorithm is applied to a foot-mounted inertial pedestrian navigation system. The navigation state is represented as an element of the matrix Lie group of double direct isometries, $SE_2(3)$, which describes the space encompassing position, velocity, and attitude in which the pedestrian operates. This model also includes biases from accelerometers and rate gyros, common components in inertial sensors. The design matrix for the navigation system is developed and discussed. It demonstrates how the stationary phase during a pedestrian's walk can be incorporated into the matrix as part of the external measurements used to update the system. This phase, often referred to as Zero Velocity Update [6], is shown to be a left-invariant measurement within the IEKF framework.

II. Methodology

A. System Modelling

Let point z be the centre of mass of a rigid body rotating and translating in 3D space. Let $\mathcal{F}_n$ be a, North-East-Down (NED) navigation reference frame and $\mathcal{F}_b$ be the body frame. Point w is a reference point in the inertial frame that is not moving. The position of z relative to w, resolved in the navigation frame is denoted $\mathbf{r}_n^{zw}$.

The velocity of the body with respect to $\mathcal{F}_n$, resolved in $\mathcal{F}_n$ is denoted $\mathbf{v}_n^{zw/n}$. The attitude of the body is described by DCM $\boldsymbol{C}_{nb}$. The angular velocity of the body, with respect to inertial frame, $\boldsymbol{a}$, resolved in $\mathcal{F}_b$ is denoted $\boldsymbol{\omega}_b^{ba}$. The kinematics of the body are then given by Equation (1) to (3).

$$\dot{\mathbf{C}}_{nb} = \mathbf{C}_{nb}\boldsymbol{\omega}_b^{ba^{\times}} \quad (1)$$

$$\dot{\mathbf{v}}_n^{z\omega/n} = \mathbf{a}_n^{z\omega/n/n} \quad (2)$$

$$\dot{\mathbf{r}}_n^{z\omega/n} = \mathbf{v}_n^{z\omega/n/n} \quad (3)$$

B. Continuous Time Process Model

Rate-gyro measurements that are biased and noisy, $\mathbf{u}_{\boldsymbol{b}}^{\mathbf{1}}$, and accelerometer measurements that are biased and noisy, $\mathbf{u}_{\boldsymbol{b}}^{\mathbf{2}}$ are given by Equation (4) and (5), respectively.

$$\mathbf{u}_b^1 = \boldsymbol{\omega}_b^{ba} - \mathbf{b}_b^1 - \mathbf{w}_b^1 \quad (4)$$

$$\mathbf{u}_b^2 = \mathbf{f}_{\boldsymbol{b}} - \mathbf{b}_b^2 - \mathbf{w}_b^2 \quad (5)$$

where noise $\mathbf{w}_b^1 \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_1)$ and noise $\mathbf{w}_b^2 \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_2)$. The bias $\mathbf{b}_b^1$ is modelled as a random walk as shown in Equation (6).

$$\dot{\mathbf{b}}_b^1 = \mathbf{w}_b^3 \quad (6)$$

where noise $\mathbf{w}_b^3 \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_3)$ and $\mathbf{f}_{\boldsymbol{b}}$ is the specific force vector resolved in the body frame. The accelerometer bias $\mathbf{b}_b^2$ is also modelled as a random walk as in Equation (7).

$$\dot{\mathbf{b}}_b^2 = \mathbf{w}_b^4 \quad (7)$$

where noise $\mathbf{w}_b^4 \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_4)$. The acceleration can then be expressed as in Equation (8).

$$\dot{\mathbf{v}}_n^{\frac{zw}{n}} = \mathbf{f}_n + \mathbf{g}_n = \mathbf{C}_{nb}\mathbf{f}_n + \mathbf{g}_n$$
$$= \mathbf{C}_{nb}(\mathbf{u}_b^2 + \mathbf{b}_b^2 + \mathbf{w}_b^2) + \mathbf{g}_n \quad (8)$$

where $\mathbf{g}_n$ is the gravity vector resolved in the navigation frame. Thus, the continuous-time kinematic process model is given as Equation (9) to (11).

$$\dot{\mathbf{C}}_{nb} = \mathbf{C}_{nb}(\mathbf{u}_b^1 + \mathbf{b}_b^1 + \mathbf{w}_b^1)^{\times} (9)$$
$$\dot{\mathbf{v}}_n^{zw/n} = \mathbf{C}_{nb}(\mathbf{u}_b^2 + \mathbf{b}_b^2 + \mathbf{w}_b^2) + \mathbf{g}_n \quad (10)$$
$$\dot{\mathbf{r}}_n^{zw/n} = \mathbf{v}_n^{zw/n} \quad (11)$$

The process model can be discretized using any appropriate discretization scheme. In this case, using a Forward Euler discretization yields as in Equation (12) to (16).

$$\mathbf{C}_{nb_k} = \mathbf{C}_{nb_k-1}\exp(T(\mathbf{u}_{b_{k-1}}^1 + \mathbf{b}_{b_{k-1}}^1 + \mathbf{w}_{b_{k-1}}^1)^{\times}) \quad (12)$$
$$\mathbf{v}_n^{z_k w} = \mathbf{v}_n^{z_{k-1} w} + T(\mathbf{C}_{nb_{k-1}}(\mathbf{u}_{b_{k-1}}^2 + \mathbf{b}_{b_{k-1}}^2 + \mathbf{w}_{b_{k-1}}^2) + \mathbf{g}_n) \quad (13)$$
$$\mathbf{r}_n^{z_k w} = \mathbf{r}_n^{z_{k-1} w} + T\mathbf{v}_n^{z_{k-1} w/n} \quad (14)$$
$$\mathbf{b}_{b_k}^1 = \mathbf{b}_{b_{k-1}}^1 + T\mathbf{w}_{b_{k-1}}^3 \quad (15)$$
$$\mathbf{b}_{b_k}^2 = \mathbf{b}_{b_{k-1}}^2 + T\mathbf{w}_{b_{k-1}}^4 \quad (16)$$

where $T = t_k - t_{k-1}$ is the timestep.

Next, the pedestrian navigation states $\mathbf{C}_{nb}$, $\mathbf{v}_n^{zw/n}$, $\mathbf{r}_n^{zw}$, $\mathbf{b}_b^1$ and $\mathbf{b}_b^2$ can be placed into an element of the Lie group $SE_2(3)$ with biases as in Equation (17).

$$\mathbf{T_1} = \begin{bmatrix} \mathbf{C}_{nb} & \mathbf{v}_n & \mathbf{r}_n & \square & \square & \square \\ \square & 1 & \square & \square & \square & \square \\ \square & \square & 1 & \square & \square & \square \\ \square & \square & \square & 1 & \mathbf{b}_{\mathrm{b}}^1 & \square \\ \square & \square & \square & \square & 1 & \mathbf{b}_{\mathrm{b}}^2 \\ \square & \square & \square & \square & \square & 1 \end{bmatrix} (17)$$

C. Measurement Model

Zero Velocity Update: The available measurements are only from accelerometers and gyros that provide noisy acceleration and angular rates measurements respectively. First, the velocity measurement is inferred through a velocity sensor (detector), specifically when the foot is detected to be in stance phase, that the position of the foot is assumed to remain fixed in the navigation N-E-D frame, and therefore the measured pseudo-velocity is zero, as shown in

Figure 1. This is known as Zero Velocity Update (ZVU or ZUPT), and many variations of ZUPT have been experimented in navigation system [7], [8], [9].

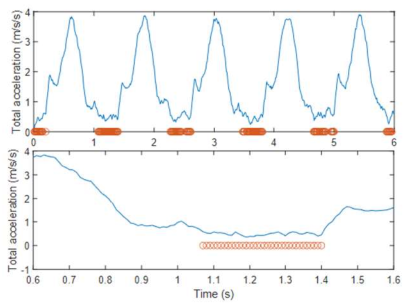

Figure 1: The red dots are detected for stance phase, where velocity is assumed to be zero

Velocity pseudo-measurement is given by Equation (18), where $\mathbf{v}^1_{n_k} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{R}_1)$.

$$\boldsymbol{y}^1_{n_k} = \mathbf{0}_{3\times1} = \mathbf{v}_n^{z_k w/n} + \mathbf{v}^1_{n_k} \quad (18)$$

These measurements can be written as a function of the navigation states contained within $\mathbf{T}$ as Equation (19).

$$\begin{bmatrix} \boldsymbol{y}^1_{n_k} \\ \square \\ \square \end{bmatrix} = \begin{bmatrix} \mathbf{v}_n^{z_k w/n} \\ \square \\ \square \end{bmatrix} + \begin{bmatrix} \mathbf{v}^1_{n_k} \\ \square \\ \square \end{bmatrix} = \begin{bmatrix} \mathbf{T}_k \begin{bmatrix} \mathbf{0} \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{v}^1_{n_k} \\ \square \\ \square \end{bmatrix} \end{bmatrix} \quad (19)$$

where the measurement model for velocity is $\boldsymbol{y}^{\mathrm{L}}_k = \mathbf{X}_k \mathbf{b}_k + \mathbf{v}_k$. Therefore, the measurement model for the velocity sensor is in left-invariant form.

D. IEKF Continuous-Time Prediction

Because the measurement model for the velocity sensor is left-invariant, a left-invariant error definition will be used. Thus, the left invariant error for the lie group $\mathcal{G}_1$ is given as Equation (20).

$$\delta\mathbf{X} = \mathbf{X}^{-1}\hat{\mathbf{X}} = \begin{bmatrix} \mathbf{T}^{-1} & \mathbf{0} & \square & \square \\ \square & \mathbf{1} & -\mathbf{b}_b^1 & -\mathbf{b}_b^2 \\ \square & \square & 1 & \square \\ \square & \square & \square & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{T}} & \mathbf{0} & \square & \square \\ \square & \mathbf{1} & \hat{\mathbf{b}}_b^1 & \hat{\mathbf{b}}_b^2 \\ \square & \square & 1 & \square \\ \square & \square & \square & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{-1}\hat{\mathbf{T}} & \mathbf{0} & \square & \square \\ \square & 1 & \hat{\mathbf{b}}_b^1 - \mathbf{b}_b^1 & \hat{\mathbf{b}}_b^2 - \mathbf{b}_b^2 \\ \square & \square & 1 & \square \\ \square & \square & \square & 1 \end{bmatrix} \quad (20)$$

Thus, the left-invariant errors can be written as Equation (21) to (23).

$$\delta\mathbf{T} = \mathbf{T}^{-1}\,\hat{\mathbf{T}} \tag{21}$$

$$\delta\mathbf{b}_b^1 = \hat{\mathbf{b}}_b^1 - \mathbf{b}_b^1 \tag{22}$$

$$\delta\mathbf{b}_b^2 = \hat{\mathbf{b}}_b^2 - \mathbf{b}_b^2 \tag{23}$$

Similarly to before, the left-invariant error for the pedestrian navigation states defined on $SE_2(3)$ can be written by expanding the left-invariant given by Equation (21). Expanding yields,

$$\delta\mathbf{T} = \mathbf{T}^{-1}\,\hat{\mathbf{T}} = \begin{bmatrix} \mathbf{C}_{nb}^{\mathrm{T}} & -\mathbf{C}_{nb}^{\mathrm{T}}\mathbf{v}_n^{zw/n} & -\mathbf{C}_{nb}^{\mathrm{T}}\mathbf{r}_n^{zw} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{C}}_{nb} & \hat{\mathbf{v}}_n^{zw/n} & \hat{\mathbf{r}}_n^{zw} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{C}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb} & \mathbf{C}_{nb}^{\mathrm{T}}(\hat{\mathbf{v}}_n^{zw/n} - \mathbf{v}_n^{zw/n}) & \mathbf{C}_{nb}^{\mathrm{T}}(\hat{\mathbf{r}}_n^{zw} - \mathbf{r}_n^{zw}) \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & 1 \end{bmatrix} \tag{22}$$

Now, the left-invariant errors are defined to be as Equation (23) to (25).

$$\delta\mathbf{C} = \mathbf{C}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb} \tag{23}$$

$$\delta\mathbf{v} = \mathbf{C}_{nb}^{\mathrm{T}}(\hat{\mathbf{v}}_n^{zw/n} - \mathbf{v}_n^{zw/n}) \tag{24}$$

$$\delta\mathbf{r} = \mathbf{C}_{nb}^{\mathrm{T}}(\hat{\mathbf{r}}_n^{zw} - \mathbf{r}_n^{zw}) \tag{25}$$

-With the addition of rate gyro and accelerometer bias, the continuous-time process model is no longer group affine. As a result, like the regular EKF or Multiplicative EKF (MEKF), error propagation can be performed on each state independently. However, unlike the regular EKF, the error definition used in this propagation is unique — the left-invariant error definitions will be applied to the error propagation.

With $(\hat{\cdot})$ is defined as measurements that include estimated bias, the attitude error propagation is given Equation (26).

$$\delta\dot{\mathbf{C}} = \dot{\mathbf{C}}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb} + \mathbf{C}_{nb}^{\mathrm{T}}\dot{\hat{\mathbf{C}}}_{nb} = -(\mathbf{u}_b^1 + \mathbf{b}_b^1 + \mathbf{w}_b^1)^{\times}\mathbf{C}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb} + \mathbf{C}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb}(\mathbf{u}_b^1 + \hat{\mathbf{b}}_b^1)^{\times} = \mathbf{C}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb}\left(\mathbf{u}_b^1 + \hat{\mathbf{b}}_b^1\right)^{\times} - \left(\mathbf{u}_b^1 + \mathbf{b}_b^1 + \mathbf{w}_b^1\right)^{\times}\mathbf{C}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb} = \mathbf{C}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb}\left(\mathbf{u}_b^{1\times} + \hat{\mathbf{b}}_b^{1\times}\right) - \left(\mathbf{u}_b^{1\times} + \mathbf{b}_b^{1\times} + \mathbf{w}_b^{1\times}\right)\mathbf{C}_{nb}^{\mathrm{T}}\hat{\mathbf{C}}_{nb} = \delta\mathbf{C}\left(\mathbf{u}_b^{1\times} + \hat{\mathbf{b}}_b^{1\times}\right) - \left(\mathbf{u}_b^{1\times} + \mathbf{b}_b^{1\times} + \mathbf{w}_b^{1\times}\right)\delta\mathbf{C} \tag{26}$$

Now, Equation (26) can be linearised by letting $\delta\mathbf{C} \approx 1 + \delta\boldsymbol{\xi}^{\emptyset\times}$, $\mathbf{b}_b^1 = \hat{\mathbf{b}}_b^1 - \delta\mathbf{b}_b^1$ and $\mathbf{w}_b^1 = \delta\mathbf{w}_b^1$ as shown in Equation (27). Neglecting product terms and cancelling necessary terms yields, Equation (27) is shown as Equation (28).

$$\delta\dot{\boldsymbol{\xi}}^{\emptyset\times} = \left(\mathbf{1} + \delta\boldsymbol{\xi}^{\emptyset\times}\right)\left(\mathbf{u}_b^{1\times} + \hat{\mathbf{b}}_b^{1\times}\right) - \left(\mathbf{u}_b^{1\times} + \hat{\mathbf{b}}_b^{1\times} - \delta\mathbf{b}_b^{1\times} + \delta\mathbf{w}_b^{1\times}\right)\left(\mathbf{1} + \delta\boldsymbol{\xi}^{\emptyset\times}\right) = \mathbf{u}_b^{1\times} + \hat{\mathbf{b}}_b^{1\times}) + \delta\boldsymbol{\xi}^{\emptyset\times}\left(\mathbf{u}_b^{1\times} + \hat{\mathbf{b}}_b^{1\times}\right) - \left(\mathbf{u}_b^{1\times} + \hat{\mathbf{b}}_b^{1\times} - \delta\mathbf{b}_b^{1\times} + \delta\mathbf{w}_b^{1\times}\right) - \left(\mathbf{u}_b^{1\times} + \hat{\mathbf{b}}_b^{1\times} - \delta\mathbf{b}_b^{1\times} + \delta\mathbf{w}_b^{1\times}\right)\delta\boldsymbol{\xi}^{\emptyset\times} \tag{27}$$

$$\delta\dot{\boldsymbol{\xi}}^{\emptyset\times} = \delta\boldsymbol{\xi}^{\emptyset\times}\mathbf{u}_b^{1\times} + \delta\boldsymbol{\xi}^{\emptyset\times}\hat{\mathbf{b}}_b^{1\times} + \delta\mathbf{b}_b^{1\times} - \delta\mathbf{w}_b^{1\times} - \mathbf{u}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset\times} - \hat{\mathbf{b}}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset\times} = \delta\boldsymbol{\xi}^{\emptyset\times}\mathbf{u}_b^{1\times} - \mathbf{u}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset\times} + \delta\boldsymbol{\xi}^{\emptyset\times}\hat{\mathbf{b}}_b^{1\times} - \hat{\mathbf{b}}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset\times} + \delta\mathbf{b}_b^{1\times} - \delta\mathbf{w}_b^{1\times} \tag{28}$$

To simplify, the identity as shown in Equation (29) must be applied, where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

$$\mathbf{u}^\times\mathbf{v}^\times - \mathbf{v}^\times\mathbf{u}^\times = (\mathbf{u}^\times\mathbf{v})^\times \tag{29}$$

Applying Equation (29) to the term $\delta\boldsymbol{\xi}^{\emptyset\times}\mathbf{u}_b^{1\times} - \mathbf{u}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset\times}$ and $\delta\boldsymbol{\xi}^{\emptyset\times}\hat{\mathbf{b}}_b^{1\times} - \hat{\mathbf{b}}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset\times}$ yields,

$$\delta\boldsymbol{\xi}^{\emptyset\times}\mathbf{u}_b^{1\times} - \mathbf{u}_b^{1\times}\boldsymbol{\delta\xi}^{\emptyset\times} = -\left(\mathbf{u}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset}\right)^\times \tag{30}$$

$$\delta\boldsymbol{\xi}^{\emptyset\times}\hat{\mathbf{b}}_b^{1\times} - \hat{\mathbf{b}}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset\times} = -\left(\hat{\mathbf{b}}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset}\right)^\times \tag{31}$$

Thus, after uncrossing both sides, the full linearised attitude error dynamics, Equation (28) can be written as Equation (32).

$$\delta\dot{\boldsymbol{\xi}}^{\emptyset\times} = -\left(\mathbf{u}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset}\right)^\times - \left(\hat{\mathbf{b}}_b^{1\times}\delta\boldsymbol{\xi}^{\emptyset}\right)^\times + \delta\mathbf{b}_b^{1\times} - \delta\mathbf{w}_b^{1\times} = \left(-\mathbf{u}_b^1 - \hat{\mathbf{b}}_b^1\right)^\times\delta\boldsymbol{\xi}^{\emptyset} + \delta\mathbf{b}_b^{1\times} - \delta\mathbf{w}_b^{1\times} \tag{32}$$

Letting $\delta\mathbf{b}_{\mathbf{b}}^1 = \delta\boldsymbol{\xi}^{\mathbf{b}^1}$ yields,

$$\delta\dot{\boldsymbol{\xi}}^{\emptyset} = \mathbf{A_1}\delta\boldsymbol{\xi} - \delta\mathbf{w}_{\mathbf{b}}^{\mathbf{1}} \tag{33}$$

where the matrix $\mathbf{A}_1$ is given by,

$$\mathbf{A}_1 = \left[\left(-\mathbf{u}_b^1 - \hat{\mathbf{b}}_b^1\right)^\times \mathbf{0} \quad \mathbf{0} \quad \mathbf{1} \quad \mathbf{0}\right] \tag{34}$$

The velocity error propagation is given by Equation (35).

$$\delta\dot{\mathbf{v}} = \dot{\mathbf{C}}_{nb}^{\mathrm{T}}\left(\hat{\mathbf{v}}_n^{zw/n} - \mathbf{v}_n^{zw/n}\right) + \mathbf{C}_{nb}^{\mathrm{T}}\left(\dot{\hat{\mathbf{v}}}_n^{zw/n} - \dot{\mathbf{v}}_n^{zw/n}\right) = -\left(\mathbf{u}_b^1 + \mathbf{b}_b^1 + \mathbf{w}_b^1\right)^\times\mathbf{C}_{nb}^{\mathrm{T}}\left(\hat{\mathbf{v}}_n^{zw/n} - \mathbf{v}_n^{zw/n}\right) + \mathbf{C}_{nb}^{\mathrm{T}}\left(\left(\hat{\mathbf{C}}_{\mathbf{nb}}\left(\mathbf{u}_b^2 + \hat{\mathbf{b}}_b^2\right) + \mathbf{g}_n\right) - \left(\mathbf{C}_{nb}\left(\mathbf{u}_b^2 + \right.\right.\right.$$

$$\mathbf{b}_b^2 + \mathbf{w}_b^2) + \mathbf{g}_n)\big) = -\big(\mathbf{u}_b^1 + \mathbf{b}_b^1 + \mathbf{w}_b^1\big)^{\times}\delta\mathbf{v} + \delta\mathbf{C}\big(\mathbf{u}_b^2 + \hat{\mathbf{b}}_b^2\big) - \big(\mathbf{u}_b^2 + \mathbf{b}_b^2 + \mathbf{w}_b^2\big) \tag{35}$$

Next, replacing $\delta\mathbf{v} = \mathbf{J}\delta\boldsymbol{\xi}^{\mathbf{v}}$, $\mathbf{b}_b^1 = \hat{\mathbf{b}}_b^1 - \delta\mathbf{b}_b^1, \mathbf{b}_b^2 = \hat{\mathbf{b}}_b^2 - \delta\mathbf{b}_b^2$ and linearising (using Equation (1) and (6)) by letting $\mathbf{J} \approx \mathbf{1}$, $\delta\mathbf{C} \approx \mathbf{1} + \delta\boldsymbol{\xi}^{\emptyset\times}$, $\mathbf{w}_b^1 = \delta\mathbf{w}_b^1$ and $\mathbf{w}_b^2 = \delta\mathbf{w}_b^2$ yields,

$$\delta\dot{\boldsymbol{\xi}}^{\mathbf{v}} = -\big(\mathbf{u}_b^1 + \hat{\mathbf{b}}_b^1 - \delta\mathbf{b}_b^1 + \delta\mathbf{w}_b^1\big)^{\times}\delta\boldsymbol{\xi}^{\mathbf{v}} + \Big(\mathbf{1} + \delta\boldsymbol{\xi}^{\emptyset\times}\Big)\big(\mathbf{u}_{\mathbf{b}}^2 + \hat{\mathbf{b}}_{\mathbf{b}}^2\big) - \big(\mathbf{u}_b^2 + \hat{\mathbf{b}}_b^2 - \delta\mathbf{b}_b^2 + \delta\mathbf{w}_b^2\big) \tag{36}$$

Neglecting higher order terms yields,

$$\delta\dot{\boldsymbol{\xi}}^{\mathbf{v}} = \big(-\mathbf{u}_b^1 - \hat{\mathbf{b}}_b^1\big)^{\times}\delta\boldsymbol{\xi}^{\mathbf{v}} + \big(-\mathbf{u}_b^2 - \hat{\mathbf{b}}_b^2\big)^{\times}\delta\dot{\boldsymbol{\xi}}^{\emptyset} + \delta\mathbf{b}_b^2 - \delta\mathbf{w}_b^2 \tag{37}$$

Letting $\delta\mathbf{b}_{\mathrm{b}}^2 = \delta\boldsymbol{\xi}^{\mathbf{b}^2}$ yields,

$$\delta\dot{\boldsymbol{\xi}}^{\mathbf{v}} = \mathbf{A}_2\delta\boldsymbol{\xi} - \delta\mathbf{w}_b^2 \tag{38}$$

where the $\mathbf{A}_2$ is given by

$$\mathbf{A}_2 = \Big[-\big(\mathbf{u}_b^2 + \hat{\mathbf{b}}_b^2\big)^{\times} \; -\big(\mathbf{u}_b^1 + \hat{\mathbf{b}}_b^1\big)^{\times} \; \mathbf{0} \quad \mathbf{0} \quad \mathbf{1}\Big] \tag{39}$$

Next, the continuous-time position error dynamics are given by Equation (40).

$$\delta\dot{\mathbf{r}} = \dot{\mathbf{C}}_{nb}^{\mathrm{T}}(\hat{\mathbf{r}}_n^{zw} - \mathbf{r}_n^{zw}) + \mathbf{C}_{nb}^{\mathrm{T}}\big(\dot{\hat{\mathbf{r}}}_n^{zw} - \dot{\mathbf{r}}_n^{zw}\big) = -(\mathbf{u}_b^1 + \mathbf{b}_b^1 + \mathbf{w}_b^1)^{\times}\mathbf{C}_{nb}^{\mathbf{T}}(\hat{\mathbf{r}}_n^{zw} - \mathbf{r}_n^{zw}) + \mathbf{C}_{nb}^{\mathrm{T}}\big(\dot{\hat{\mathbf{r}}}_n^{zw} - \dot{\mathbf{r}}_n^{zw}\big) = -(\mathbf{u}_b^1 + \mathbf{b}_b^1 + \mathbf{w}_b^1)^{\times}\delta\mathbf{r} + \delta\mathbf{v} = -\big(\mathbf{u}_b^1 + \hat{\mathbf{b}}_b^1 - \delta\mathbf{b}_b^1 + \boldsymbol{\delta}\mathbf{w}_b^1\big)^{\times}\mathbf{J}\delta\boldsymbol{\xi}^{\mathbf{r}} + \mathbf{J}\delta\boldsymbol{\xi}^{\mathbf{v}} \tag{40}$$

Linearizing by letting $\mathbf{J} \approx \mathbf{1}, \mathbf{w}_b^1 = \delta\mathbf{w}_b^1$ and neglecting higher order terms yields,

$$\delta\dot{\boldsymbol{\xi}}^{\mathrm{r}} = -\big(\mathbf{u}_b^1 + \hat{\mathbf{b}}_b^1\big)^{\times}\delta\boldsymbol{\xi}^{\mathrm{r}} + \delta\boldsymbol{\xi}^{\mathrm{v}} = \mathbf{A}_3\delta\boldsymbol{\xi} \tag{41}$$

where the matrix $\mathbf{A_3}$ is given by

$$\mathbf{A_3} = \big[\mathbf{0} \quad \mathbf{1} \quad -\big(\mathbf{u}_b^1 + \hat{\mathbf{b}}_b^1\big)^{\times} \quad \mathbf{0} \quad \mathbf{0}\big] \tag{42}$$

Lastly, letting $\delta\mathbf{b}_{\mathrm{b}} = \delta\boldsymbol{\xi}^{\mathrm{b}}$, the bias error propagation is,

$$\delta\boldsymbol{\xi}^{\mathbf{b}^1} = \dot{\hat{\mathbf{b}}}_b^1 - \dot{\mathbf{b}}_b^1 = -\mathbf{w}_b^3 = -\mathbf{w}_b^4 \tag{43}$$

$$\delta\boldsymbol{\xi}^{\mathbf{b}^2} = \dot{\hat{\mathbf{b}}}_b^2 - \dot{\mathbf{b}}_b^2 = -\mathbf{w}_b^4 = -\mathbf{w}_b^5 \tag{44}$$

The full linearised **A** matrix is then written by concatenating each of the previous matrices vertically. The linearized error dynamics are given as Equation (45).

$$\delta\dot{\boldsymbol{\xi}} = \underbrace{\begin{bmatrix} -(\mathbf{u}_b^1+\hat{\mathbf{b}}_b^1)^\times & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ -(\mathbf{u}_b^2+\hat{\mathbf{b}}_b^2)^\times & -(\mathbf{u}_b^1+\hat{\mathbf{b}}_b^1)^\times & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & -(\mathbf{u}_b^1+\hat{\mathbf{b}}_b^1)^\times & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\boldsymbol{A}} \delta\boldsymbol{\xi} - \underbrace{\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}}_{\mathbf{L}} \delta\mathbf{w} \quad (45)$$

The continuous time linearised process model is then discretized using any appropriate method. In this case, a standard forward Euler discretization scheme was selected, and the discrete time process model Jacobians are written as,

$$\mathbf{A}_k = \mathbf{1} + T\mathbf{A}\,,\ \mathbf{L}_k = T\mathbf{L} \quad (46)$$

Then, the prediction step is carried out as,

$$\check{\mathbf{X}}_k = \mathbf{F}_{k-1}(\hat{\mathbf{X}}_{k-1}, \mathbf{u}_{k-1})$$
$$\check{\mathbf{P}}_k = \mathbf{A}_{k-1}\hat{\mathbf{P}}_{k-1}\mathbf{A}_{k-1}^{\mathrm{T}} + \mathbf{L}_{k-1}\mathbf{Q}_{k-1}\mathbf{L}_{k-1}^{\mathrm{T}} \quad (47)$$

E. IEKF Discrete-Time Correction

The left-invariant correction is used in the form of

$$\hat{\mathbf{T}} = \check{\mathbf{T}}\exp(-(\mathbf{K}_k\,\mathbf{z}_k)^\wedge) \quad (48)$$

where $\mathbf{K}_k$ is the Kalman gain and $\mathbf{z}_k$ is the innovation term, and an additive correction for the bias state (if there exist such measurements). The velocity measurement model given by Equation (18) is left-invariant, and therefore a left-invariant innovation of the form $\boldsymbol{z}_k = [\check{\mathbf{X}}_1^{-1}(\mathbf{y}_k - \check{\mathbf{y}}_k)]$ is used.

The full innovation term is then given by Equation (49).

$$\boldsymbol{z}_k = [\check{\mathbf{T}}_k^{-1}(\mathbf{y}_{n_k}^1 - \check{\mathbf{y}}_{n_k}^1)] \quad (49)$$

The Kalman gain, $\mathbf{K}_k$ is computed through the standard Kalman Filter equations given by,

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$$\mathbf{S}_k = \mathbf{H}_k\, \check{\mathbf{P}}_k\, \mathbf{H}_k^{\mathrm{T}} + \mathbf{M}_k\, \mathbf{R}_k\, \mathbf{M}_k^{\mathrm{T}} \tag{50}$$

$$\mathbf{K}_k = \check{\mathbf{P}}_k\, \mathbf{H}_k^{\mathrm{T}}\, \mathbf{S}_k^{-1} \tag{51}$$

The correction is computed by,

$$\hat{\mathbf{T}} = \check{\mathbf{T}} \exp(-(\mathbf{K}_k \mathbf{z}_k)^\wedge) \tag{52}$$

with the covariance is computed using Equation (53).

$$\hat{\mathbf{P}}_k = (\mathbf{1} - \mathbf{K}_k \mathbf{H}_k)\, \check{\mathbf{P}}_k\, (\mathbf{1} - \mathbf{K}_k \mathbf{H}_k)^{\mathrm{T}} + \mathbf{K}_k\, \mathbf{M}_k\, \mathbf{R}_k\, \mathbf{M}_k^{\mathrm{T}}\, \mathbf{K}_k^{\mathrm{T}} \tag{53}$$

To compute the matrices $\mathbf{H}_k$ and $\mathbf{M}_k$, the measurement models must be linearized. However, in the linearization, it is key that the correct error term be used. In this case, the left invariant error definition must be used.

Recall that the left-invariant errors are given by,

$$\delta \mathbf{C}_k = \mathbf{C}_{\mathrm{nb}_k}^{\mathrm{T}} \check{\mathbf{C}}_{\mathrm{nb}_k} \tag{54}$$

$$\delta \mathbf{v}_k = \mathbf{C}_{\mathrm{nb}_k}^{\mathrm{T}} (\check{\mathbf{v}}_n^{z_k w/n} - \mathbf{v}_n^{z_k w/n}) \tag{55}$$

$$\delta \mathbf{r}_k = \mathbf{C}_{\mathrm{nb}_k}^{\mathrm{T}} (\check{\mathbf{r}}_n^{z_k w} - \mathbf{r}_n^{z_k w}) \tag{56}$$

Velocity Correction: First, the measurement models for the velocity sensor must be linearized. Equation (57) shows the expanding equation of Equation (49) yields. Next, to linearize, let $\delta \check{\mathbf{T}}_k^{L^{-1}} \approx \left(\mathbf{1} - \boldsymbol{\delta} \check{\boldsymbol{\xi}}_k^{\boldsymbol{T}^\wedge}\right)$ and $\mathbf{v}_{n_k}^1 = \delta \mathbf{v}_{n_k}^1$. Then, $\mathbf{z}_k$ is written as Equation (58).

$$\check{\mathbf{T}}_k^{-1}\left(\begin{bmatrix}\mathbf{y}_{n_k}^1\\0\\0\end{bmatrix} - \begin{bmatrix}\check{\mathbf{y}}_{n_k}^1\\0\\0\end{bmatrix}\right) = \check{\mathbf{T}}_k^{-1}\left(\mathbf{T}_k\begin{bmatrix}\mathbf{0}\\1\\0\end{bmatrix} + \begin{bmatrix}\mathbf{v}_{n_k}^1\\0\\0\end{bmatrix} - \check{\mathbf{T}}_k\begin{bmatrix}\mathbf{0}\\1\\0\end{bmatrix}\right) = \delta\check{\mathbf{T}}_k^{-1}\begin{bmatrix}\mathbf{0}\\1\\0\end{bmatrix} + \check{\mathbf{T}}_k^{-1}\begin{bmatrix}\mathbf{v}_{n_k}^1\\0\\0\end{bmatrix} - \begin{bmatrix}\mathbf{0}\\1\\0\end{bmatrix} \tag{57}$$

$$\mathbf{z}_k \approx \left(\mathbf{1} - \delta\check{\boldsymbol{\xi}}_k^{T^{\wedge}}\right)\begin{bmatrix}\mathbf{0}\\1\\0\end{bmatrix} + \check{\mathbf{T}}_k^{-1}\begin{bmatrix}\delta\mathbf{v}_{n_k}^1\\0\\0\end{bmatrix} - \begin{bmatrix}\mathbf{0}\\1\\0\end{bmatrix} = -\delta\check{\boldsymbol{\xi}}_k^{\boldsymbol{T}^{\wedge}}\begin{bmatrix}\mathbf{0}\\1\\0\end{bmatrix} + \check{\mathbf{T}}_k^{-1}\begin{bmatrix}\delta\mathbf{v}_{n_k}^1\\0\\0\end{bmatrix} = -\begin{bmatrix}\delta\check{\boldsymbol{\xi}}_k^{\phi\times} & \delta\check{\boldsymbol{\xi}}_k^{v} & \delta\check{\boldsymbol{\xi}}_k^{r}\\ \mathbf{0} & 0 & 0\\ \mathbf{0} & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{0}\\1\\0\end{bmatrix} + \check{\mathbf{T}}_k^{-1}\begin{bmatrix}\delta\mathbf{v}_{n_k}^1\\0\\0\end{bmatrix} = \mathbf{H}\delta\check{\boldsymbol{\xi}}^{\boldsymbol{T}} + \check{\mathbf{T}}_k^{-1}\begin{bmatrix}\delta\mathbf{v}_{n_k}^1\\0\\0\end{bmatrix} \tag{58}$$

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where

$$\mathbf{H} = \begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 0 \end{bmatrix} \tag{59}$$

Note that the bottom row of Equation (59) is only zeros. Therefore, Equation (58) can be written as Equation (60).

$$\boldsymbol{z}_k = \mathbf{H}\delta\check{\boldsymbol{\xi}}_k^T + \mathbf{M}_k\delta\mathbf{v}_{n_k}^1 \tag{60}$$

where $\mathbf{H} = [\mathbf{0} \quad \mathbf{-1} \quad \mathbf{0}]$ (61)

The full linearized $\mathbf{H}_{\boldsymbol{k}}$ matrix is given as,

$$\mathbf{H}_{\boldsymbol{k}} = [\mathbf{0} \quad \mathbf{0} \quad \mathbf{-1} \quad \mathbf{0} \quad \mathbf{0}] \tag{62}$$

The measurement model Jacobian with respect to the noise, $\mathbf{M}_{\boldsymbol{k}}$, is written as,

$$\mathbf{M}_k = \check{\mathbf{C}}_{nb}^{\mathrm{T}} \tag{63}$$

III. Results and Discussions

For a foot-mounted inertial pedestrian navigation IEKF, the full linearised error dynamics for $SE_2(3)$ with gyro bias and accelerometer bias are successfully derived and can be written as Equation (64) with full linearised innovation $\boldsymbol{z}$ using velocity sensor. That can be written as Equation (65).

$$\delta\dot{\boldsymbol{\xi}} = \underbrace{\begin{bmatrix} -\left(\mathbf{u}_\mathrm{b}^1 + \hat{\mathbf{b}}_\mathrm{b}^1\right)^\times & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ -\left(\mathbf{u}_\mathrm{b}^2 + \hat{\mathbf{b}}_\mathrm{b}^2\right)^\times & -\left(\mathbf{u}_\mathrm{b}^1 + \hat{\mathbf{b}}_\mathrm{b}^1\right)^\times & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & -\left(\mathbf{u}_\mathrm{b}^1 + \hat{\mathbf{b}}_\mathrm{b}^1\right)^\times & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\boldsymbol{A}} \delta\boldsymbol{\xi} + \underbrace{\begin{bmatrix} \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{bmatrix}}_{\mathbf{L}} \delta\mathbf{w} \tag{64}$$

$$\boldsymbol{z} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\boldsymbol{H}} \delta\boldsymbol{\xi} + \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \check{\mathbf{C}}_{nb}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\boldsymbol{M}} \delta\boldsymbol{v} \tag{65}$$

It is important to note that the Jacobians for the process model, $\boldsymbol{A}$, derived using invariant error definitions by Equation (64), rely solely on the measurements from rate gyros and accelerometers, $\mathbf{u}_b^{\mathbf{1}}$ and $\mathbf{u}_b^{\mathbf{2}}$, along with the bias estimates. In contrast, the Jacobians for the Extended Kalman Filter (EKF) [10] depend not only on the rate-gyros and accelerometers measurements and bias estimates, but also on the estimated attitude, $\hat{\mathbf{C}}_{\mathrm{ab}}$. Additionally, the Jacobian for the measurement model, $\boldsymbol{H}$, given in Equation (65), is not influenced by the attitude estimates, unlike the measurement model $\mathbf{H}^*$ discussed in [11]. As a result, the IEKF Jacobians have less dependence on state estimates compared to those from the EKF, which gives the IEKF an advantage over the MEKF. This is because poor state estimates can lead to inaccurate Jacobians, undermining performance. The invariant framework used in the IEKF makes it less sensitive to initialization errors and provides better performance.

IV. Conclusion

This article revisits the theory of Lie Groups and the Invariant Extended Kalman Filter (IEKF) and applies these concepts to design an observer for a foot-mounted inertial pedestrian navigation system. Specifically, it employs the IEKF framework with elements from a matrix Lie Group to derive a design matrix, which contains the Jacobians for the process model and the measurement model. This approach aims to create a robust and accurate navigation system. The derived Jacobians have reduced dependence on state estimates, as inaccurate estimates can lead to faulty Jacobians and, ultimately, unreliable navigation. The paper derives the complete linearized error dynamics for $SE_2(3)$, considering both gyro bias and accelerometer bias, offering a comprehensive and precise depiction of the system's behavior.

Next, the full linearized measurement model for the velocity sensor is developed, capitalizing on the fact that there are times during the stance phase when a pedestrian's velocity should be theoretically zero. This measurement model is

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demonstrated to be left invariant, allowing it to be smoothly integrated into the IEKF framework. By leveraging this left invariance, the IEKF can effectively update the system's states and correct any sensor measurement errors. Ultimately, this method should enhance the accuracy and robustness of the foot-mounted inertial pedestrian navigation system.

Future work will include simulation and experimental studies on the foot-mounted pedestrian navigation system. Additional aided measurements, like magnetometer readings for attitude estimation, particularly heading, will be explored and tested within the IEKF framework, allowing for further investigation into the consistency and observability of the states.

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